TIME SERIES 501

Lesson 7: Time Series and Signal Transformations

Sine Wave

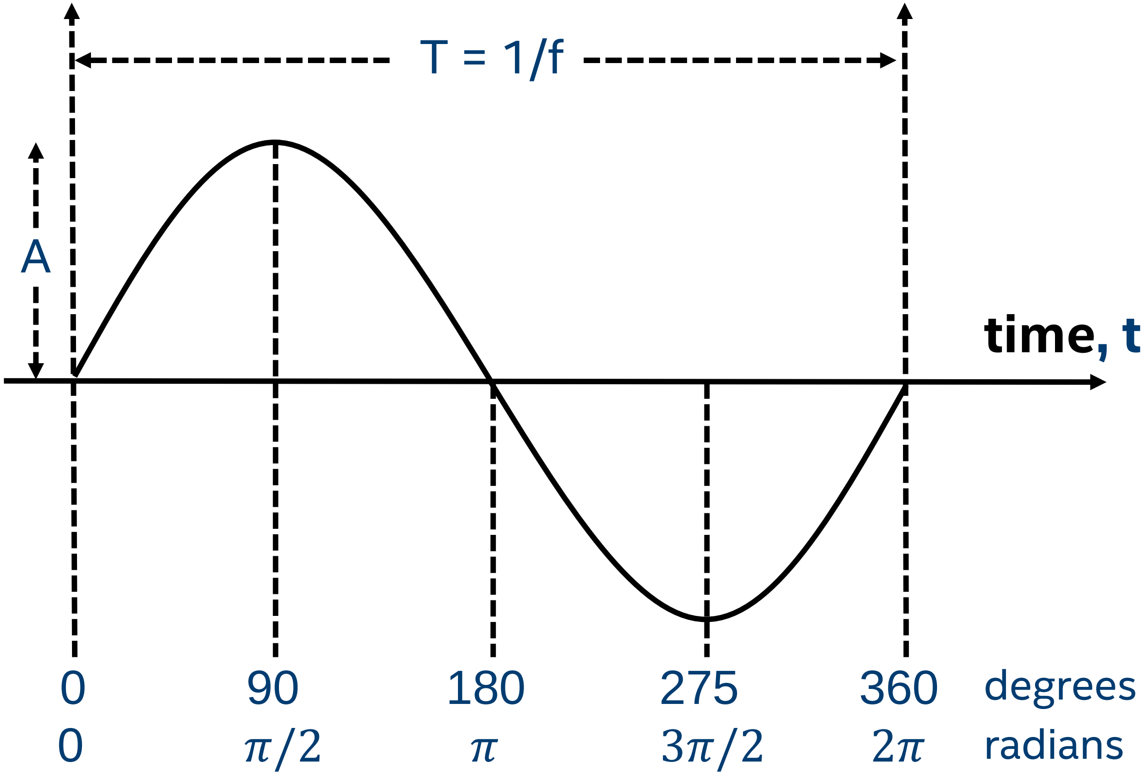
- A mathematical curve that describes smooth periodic oscillations.

- It is continuous and is described by:

$$y(t) = Asin(2\pi ft + \varphi)$$

where A is the amplitude, f is the frequency, and φ is the phase in radians

Sine Wave



A = amplitude
f = frequency
T = period

Fourier Transform (FT)

- Decomposes a time signal into the sum of sines with varying amplitudes, frequencies, and phases.

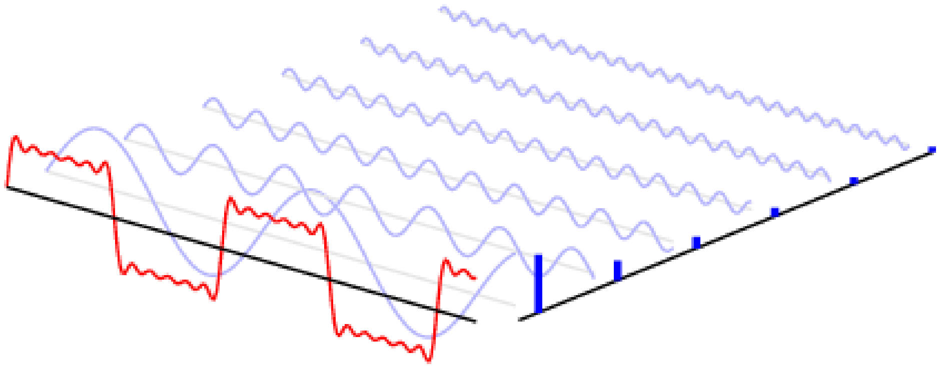
- It is also known as the frequency domain representation of the original signal.

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} dx$$

where ξ represents the frequency

Fourier Transform (FT)

The image below shows the decomposition of an almost square wave into several sines of different amplitude, phase, and frequency.



Properties of Fourier Transform (FT)

	Time Domain	Frequency Domain
• Linearity	$ax(t) + by(t)$	$aX(\xi) + bY(\xi)$
• Time Shift	$x(t - t_0)$	$e^{-j\xi t_0}X(\xi)$
• Frequency Shift	$e^{j\xi_0 t}x(t)$	$X(\xi - \xi_0)$
• Scaling	$x(at)$	$\frac{1}{\lvert a \rvert}X\left(\frac{\xi}{a}\right)$
• Differentiation in Time	$\frac{d}{dx}x(t)$	

Properties of Fourier Transform (FT)

	Time Domain	Frequency Domain
• Differentiation in Freq.	$-jtx(t)$	$\frac{d}{d\xi}X(\xi)$
• Integration	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{X(\xi)}{j\xi} + \pi X(0)\delta(\xi)$
• Convolution	$\int_{-\infty}^{\infty} x(\tau)y(t-\tau)d\tau$	$X(\xi)Y(\xi)$
• Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\Gamma)Y(\xi-\Gamma)d\Gamma$
• Parseval's Theorem	$\int_{-\infty}^{\infty} \lvert x(t)\rvert dt$	$\frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(\xi)\rvert^2 d\xi$

Common Fourier Transform (FT)

Time Domain	Frequency Domain
$x(t) = \begin{cases} 1, \lvert t \rvert < T \\ 0, otherwise \end{cases}$	$X(\xi) = \dfrac{2\pi \sin(\xi T)}{\xi}$
$x(t) = \dfrac{\sin(Wt)}{\pi t}$	$X(\xi) = \begin{cases} 1, \lvert \xi \rvert < W \\ 0, otherwise \end{cases}$
$x(t) = \delta(t)$	$X(\xi) = 1$
$x(t) = 1$	$X(\xi) = 2\pi\delta(\xi)$

Common Fourier Transform (FT)

Time Domain	Frequency Domain
$x(t) = u(t)$	$X(\xi) = \frac{1}{j\xi} + \pi\delta(\xi)$
$x(t) = \delta(t - t_0)$	$X(\xi) = e^{-j\xi t_0}$
$x(t) = e^{j\xi_0 t}$	$X(\xi) = 2\pi\delta(\xi - \xi_0)$
$x(t) = \cos(\xi_0 t)$	$X(\xi) = \pi[\delta(\xi - \xi_0) + \delta(\xi + \xi_0)]$
$x(t) = \sin(\xi_0 t)$	$X(\xi) = j\pi[\delta(\xi + \xi_0) - \delta(\xi - \xi_0)]$

Inverse Fourier Transform (IFT)

The inverse of the Fourier Transform is given by:

for any real number x $$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x \xi} \, d\xi$$

Fast Fourier Transform (FFT)

- FFT is an algorithm that samples a signal over a period of time.
- This algorithm is used by computer systems when calculating the Fourier Transform of discrete samples.

Transfer Functions

A transfer function (H) describes the output value of a system for each possible input.

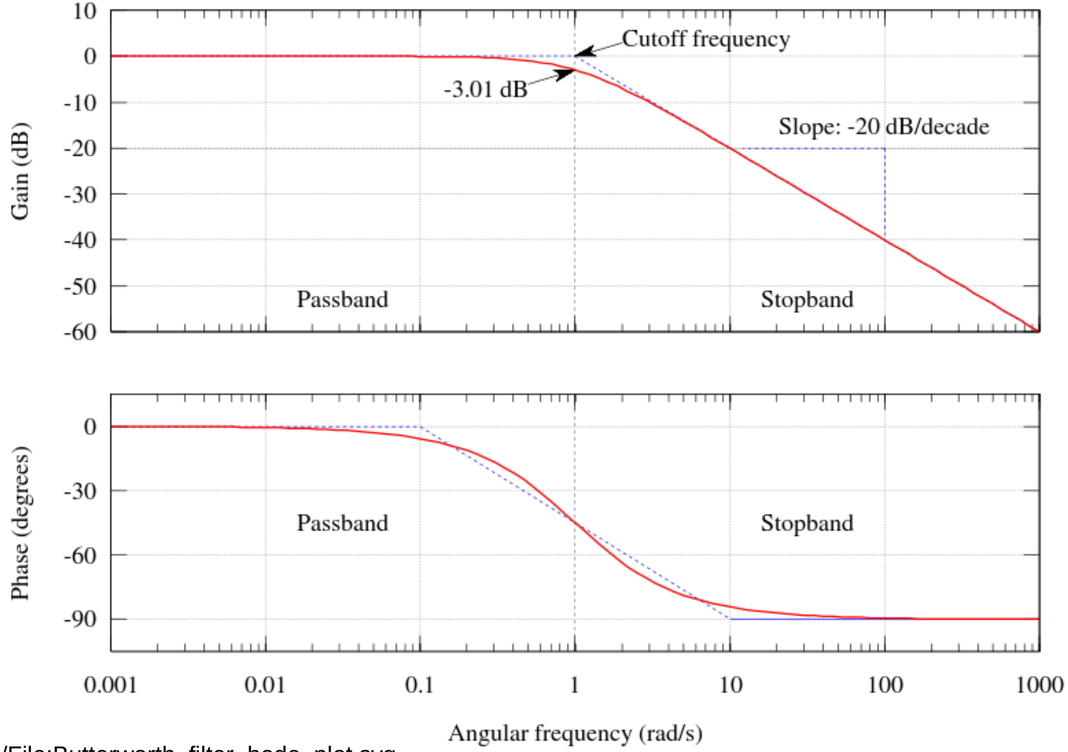
$$H(\xi) = \frac{Y(\xi)}{X(\xi)}$$

Bode Plot

- It is a graph that displays the transfer function of a system.
- It displays the amplitude (usually in decibels) of a system as a function of the frequency.
- It can also display the phase of a system as a function of the frequency.

Bode Plot

Example of a low pass Butterworth filter Bode plot of the amplitude (top) and phase (bottom)



https://commons.wikimedia.org/wiki/File:Butterworth_filter_bode_plot.svg

Filters

- A filter is a system that removes unwanted frequency components from a signal.

- Some important filters include: Chebyshev, Butterworth, Bessel, and Elliptic.

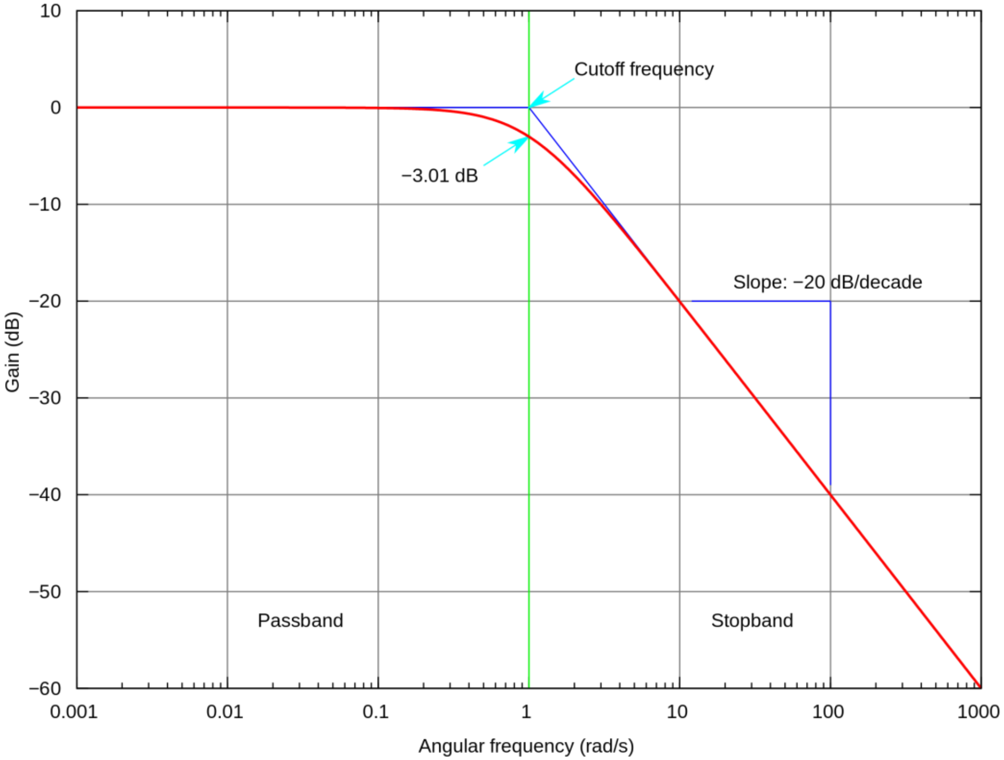
Filter Types

- Low pass
- High pass
- Band pass
- Band stop

Low-Pass Filter (LPF)

- Passes signals with a frequency lower than a certain cutoff frequency.
- Moving-average operations are a particular kind of low-pass filters.
- Low-pass filters provide a smoother form of a signal, removing the short-term fluctuations.

Low-Pass Filter (LPF)

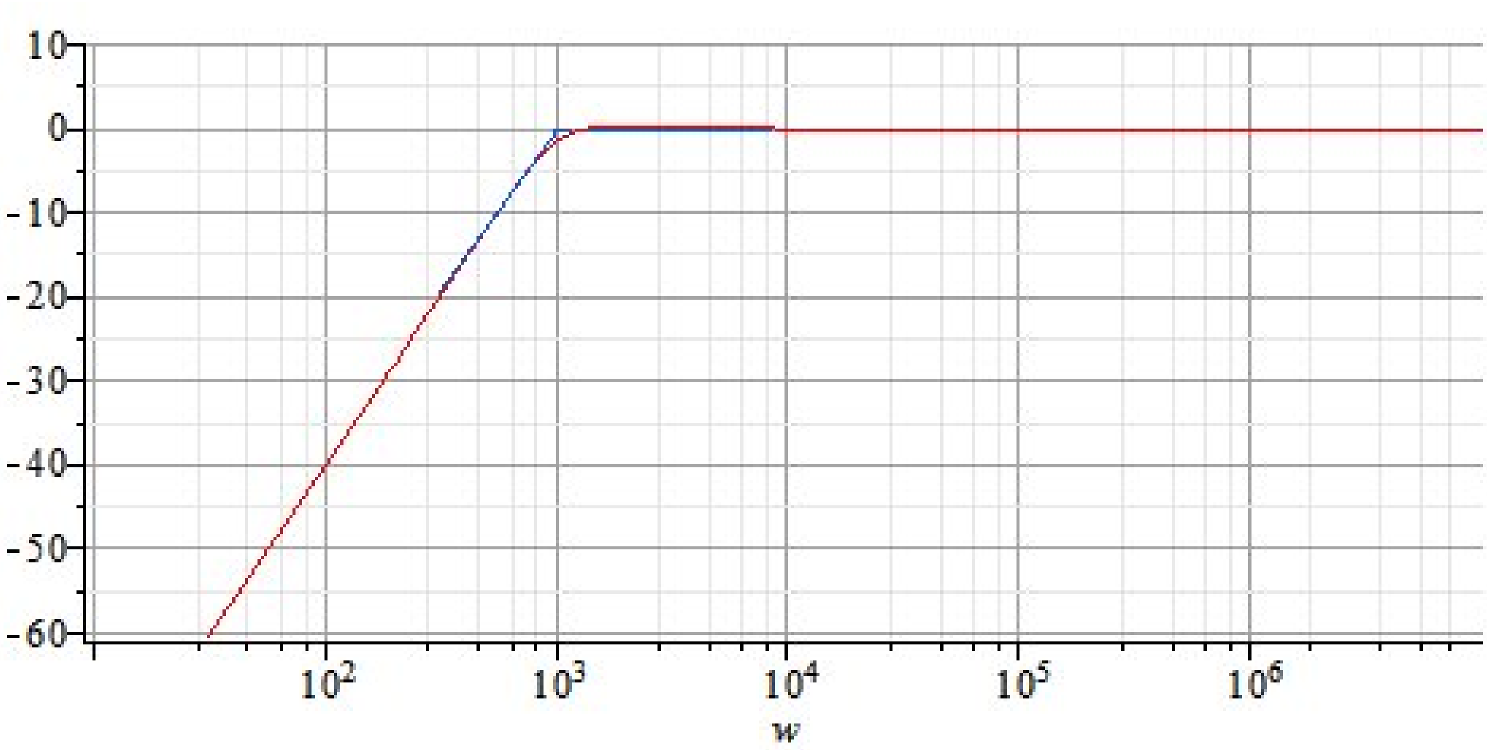


https://commons.wikimedia.org/wiki/File:Butterworth_response.svg

High-Pass Filter (HPF)

- Passes signals with a frequency higher than a certain cutoff frequency.
- Is usually modeled as a linear time-invariant system.
- High-pass filters have many uses, such as removing constants or trends.

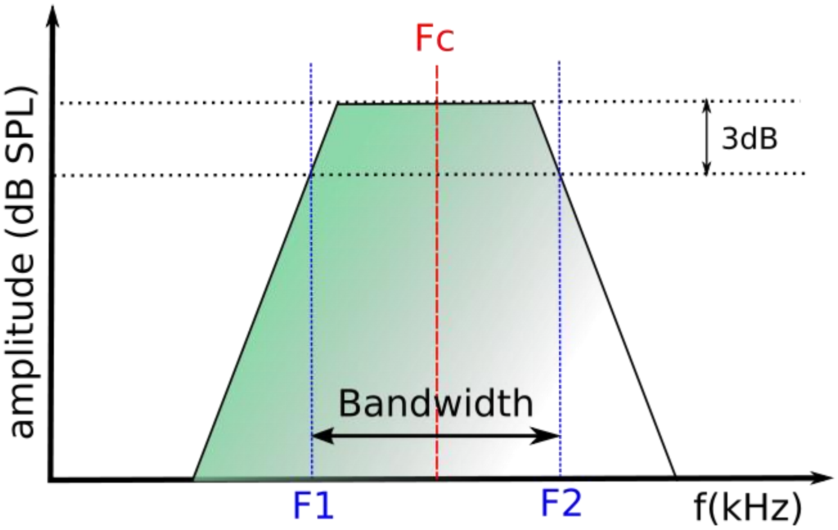
High-Pass Filter (HPF)



https://commons.wikimedia.org/wiki/File:Bode_amplitude.jpg

Band-Pass Filter (BPF)

Passes frequencies within a certain range and rejects frequencies outside that range.



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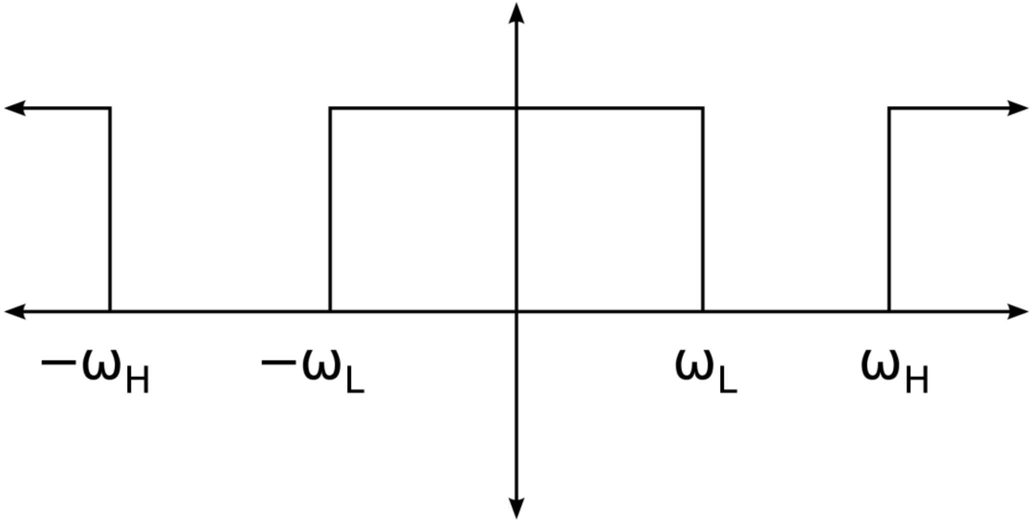
Band-Stop Filter (BSF)

Passes most filters unaltered, but attenuates those in a specific range to very low levels.



https://commons.wikimedia.org/wiki/File:Ideal_Band_Stop_Filter_Transfer_Function.svg

Butterworth Filter

Is designed to have a frequency response as flat as possible in the pass band.

$$|H(\xi)|^2 = \frac{{G_0}^2}{1 + \left(\frac{j\xi}{j\xi_c}\right)^{2n}}$$

where ξc is the cutoff frequency, G0 is the gain when the frequency is zero, and n is the order of the filter

Window Functions

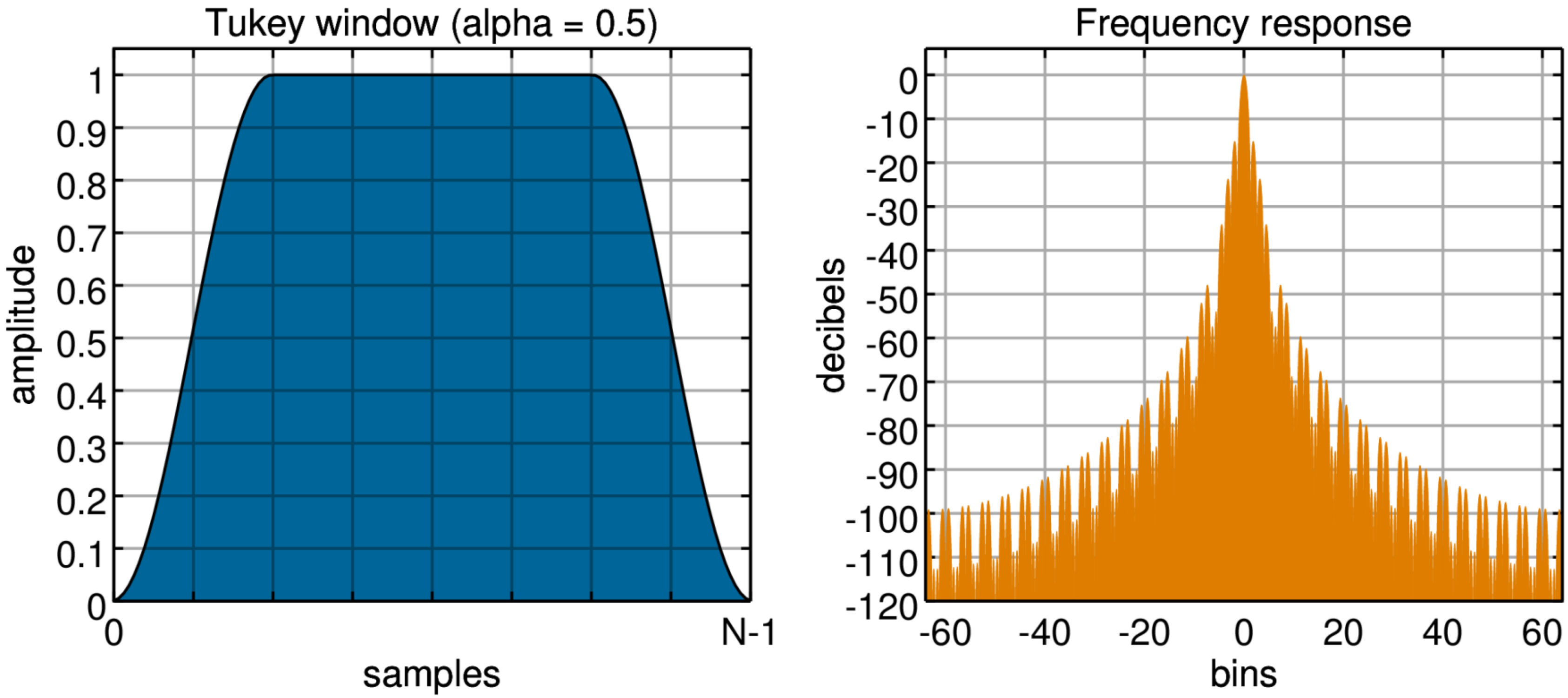
- Are functions that are zero outside a chosen interval.
- Examples include the Tukey and Hann window.

Tukey Window

Regarded as a cosine lobe of width αN/2 that is convolved with a rectangular window of width (1- α/2)N where N is the number of samples.

$$w(n) = \begin{cases} \frac{1}{2}\left[1 + cos\left(\pi\left(\frac{2n}{\alpha(N-1)} - 1\right)\right)\right], & 0 \le n < \frac{\alpha(N-1)}{2} \\ 1, & \frac{\alpha(N-1)}{2} \le n < (N-1)\left(1 - \frac{\alpha}{2}\right) \\ \frac{1}{2}\left[1 + cos\left(\pi\left(\frac{2n}{\alpha(N-1)} - \frac{2}{\alpha} + 1\right)\right)\right], & (N-1)\left(1 - \frac{\alpha}{2}\right) \le n < (N-1) \end{cases}$$

Tukey Window

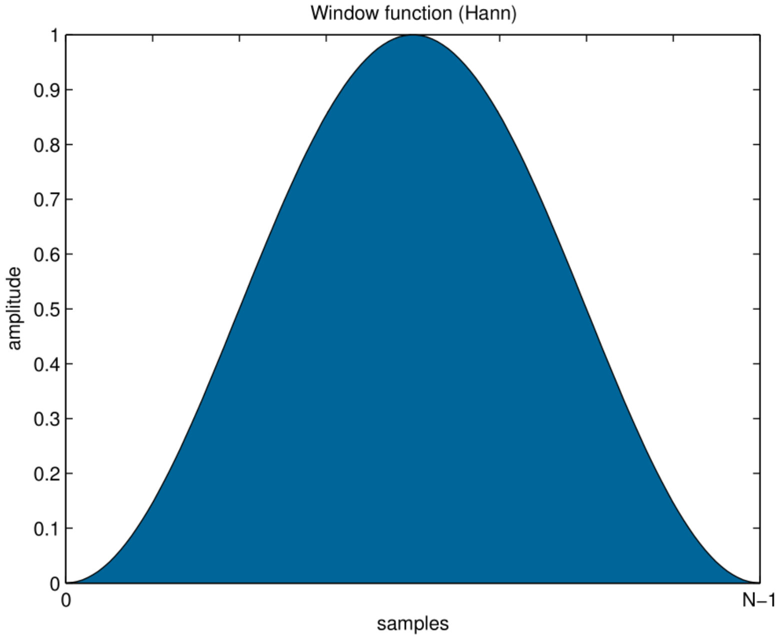


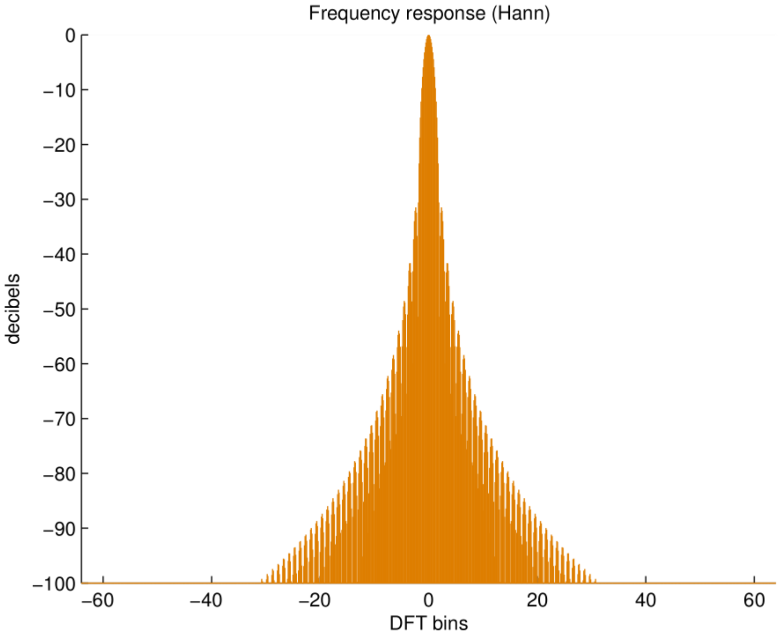
https://commons.wikimedia.org/wiki/File:Window_function_(Tukey;_alpha_%3D_0.5).png

Hann Window

- A special case of the Tukey window when $\alpha = 1$
- Also known as a raised cosine window

Hann Window





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